Stable matching with double infinity of workers and firms*

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Abstract

In this paper we analyze the existence of stable matchings in a two-sided large market in which workers are assigned to firms. The market has a continuum of workers while the set of firms is countably infinite. We show that, under certain reasonable assumptions on the preference correspondences, stable matchings not only exist but are also Pareto optimal.
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1 Introduction

Matching theory can be broadly characterized as the field of Mathematical Economics that studies the allocation of resources in "thin" two-sided markets, i.e. markets in which the goods are indivisible and heterogeneous [7]. The seminal work of Gale and Shapley ([8]) laid the ground for further developments, introducing the fundamental notion of *stability* as a solution to the problem of assigning elements of one side of the market to elements in the other one. From then on, a large body of literature grew up, motivated by the insights on the inner workings of real-world markets provided by matching theory [20]. Applications to particular instances (NRMP, kidney transplants, school admissions, etc.) spurred further interest in the field [23].

Of particular relevance for this paper are the mathematical developments in this area. Starting from the analyses of many-to-one and many-to-many matching problems, some results indicate ways in which the concept of stability can be

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extended, from the one-to-one case to those contexts (see [12], [6], [17], among many others). Interestingly, until very recently, all these advances focused on markets with a finite number of participants on both sides. Now the scope has been widened as to encompass infinities in one side of the market [1], [3].

The purpose of this work is to go still further in the direction of analyzing large markets in many-to-one settings. We adopt, for the sake of clarity, the view that one side of the market is constituted by "workers" and the other by "firms". We assume the former constitute a continuum, while the latter are countably infinite. From a mathematical point of view, it is a natural generalization of the work of [3]. Nevertheless, from an economic point of view it captures, among others, the idea of large labor markets on both sides, with perfect competition on workers side, in the sense of having "many more workers than firms", being the latter class of countably infinite cardinality.

On the one hand, double infinity of agents has already been considered in the literature on matching problems mainly in buyer/seller settings. For instance, [9] and [10] consider a continuum of agents on both sides of the assignment problem. Nevertheless, this kind of matching problem differs in many ways from ours, as for instance by assuming the transferability of utilities. In [14] each side of the market has also a continuum of agents, although under the assumption of imperfectly transferable utility functions. More recently, [19] developed a general nonlinear duality theory and applied it to matching theory with double (uncountable) infinities of agents. Their structure is also different from ours since they use a distributional approach *á la* [11] to represent the buyers and sellers. Their main assumptions are also different.

On the other hand, double infinity has interest by itself and its importance goes beyond the representation of large markets. Actually, as it is known in the literature on general equilibrium theory with infinite dimensional spaces, infinite commodity spaces arise naturally when one considers infinite time horizon or uncertainty about an infinite number of states of the world (see [18]). Thus, the aim of this paper is to include and adapt those cases to a matching problem. One of the real-life examples we can consider is that of a large market of labor with mobility along countries, provinces or states. If the same firm is located in several places (countries, provinces, states, etc) one can see it as a different firm in the same way that the same kind of physical good located in different places is considered as if it were a class of different goods ([4], pgs. 29-30). If we assume infinitely (yet countable) many places in the world to localize a firm, then it is clear that we need a more general framework than that of [3]. Let us note that this represents a real phenomena since many firms are located all over the world with labor mobility arising even in the form of illegal immigration. An indeed this is a matching problem.

Another real-world related situation arises when allocations are along time and the horizon is infinite (i.e. without a previously defined final period). For instance, let us consider a large labor market given by a mass or workers who have to decide what firm they prefer along time. Even though there is a finite set of firms over which they have to make their choices in each period, if time is unbounded, it is equivalent, from a mathematical viewpoint, to have countably infinite firms.

Alternatively, we can consider a case in which assignments are subject to uncertainty, with a countably infinite set of states of the world. In this case, one (type of) worker may choose firm f if state s_1 takes place, or she may prefer f' if state s_2 happens instead. A real life case example is when a worker has to decide whether to apply for a job in a firm belonging to an industry that could or could not be benefited by economic policies to be determined in an upcoming election. One can go a step further and add that this decision may depend upon the candidate who could win the election. For instance, the fortunes of a firm may depend on whether the election will be won by a protectionist or a pro free-trade candidate, conditioning in turn the election of the worker. Another case may arise when the worker looks for being hired by a firm on the basis of future profits that may depend on external factors like the market price of commodities. Small countries take international prices as given, which affects the return of its firms. In this case a worker may decide on the basis of (for example) the business sentiment in central countries. In all these cases, the firm to which a worker will try to match depends on the state of the world. It these are non-finite, it is again equivalent to having a countably infinite number of firms. So, apart from being mathematically interesting, this problem has economic and social relevance.

On the other hand, the consideration of a continuum of workers is also of interest on itself since many real world matching markets are large. Indeed, in the US and in Latin America there is a large market for medical doctors, students and even marriages. Since this setting generalizes the model with a continuum of workers and a finite number of firms presented in [3], extra conditions have to be imposed in order to ensure the existence of stable matchings. They amount to ask that the preferences of the firms satisfy a revealed preferences condition and a certain type of "continuity". These assumptions are close to the usual ones in General Equilibrium Theory, hinting that methods used to study the existence of equilibria in large economies with infinite dimensional commodity spaces may be adapted to this matching problem. This is the approach taken here. Of mathematical interest to our proof is the topological duality between the space of signed Borel measures on a compact metric space and the space of continuous functional on the compact metric space.

The plan of the paper is as follows. Section 2 presents the model, in particular describing its topological features. Section 3 adapts the concept of stable matching to this context. Section 4 introduces the assumptions about the preferences of the firm and states the main results of the paper, namely the existence of stable matchings, which furthermore are Pareto optimal. In the Appendix we present the proofs of these claims.

2 The model

Let us consider an economy with a continuum of workers and with infinitely (countable) many firms.[1] A measurable space $(\Theta, \sigma(\Theta))$ represents a continuum where each θ in Θ is a type of worker. Θ is assumed to be a compact metric space and $\sigma(\Theta)$ is the Borel $\sigma-$algebra over Θ.

Let $\overline{\chi}$ be the set of all non-negative measures over $(\Theta, \sigma(\Theta))$. Let $G \in \overline{\chi}$ be the particular Borel measure corresponding to the distribution of workers. That is, given any $E \in \sigma(\Theta)$, $G(E)$ is the measure of workers belonging to E. For the sake of normalization we assume that $G(\Theta) = 1$.

On the other side of the market we consider a set of firms $F := \{f_1, f_2, ..., f_{n-1}, f_n, f_{n+1}, ...\}$. Since there might be workers that remain without being matched to any firm, we add the null firm $\emptyset$. Hence, we consider $\widetilde{F} := F \cup \emptyset$.

Let $X \in \overline{\chi}$. By a slight abuse of language we call X a *subpopulation* of Θ if $X \leq G$.[2] The set of all subpopulations of Θ is $\chi \subset \overline{\chi}$. For a given $X \in \chi$, $X' \in \chi$ is said to be a subpopulation of X if $X' \leq X$. The set of all subpopulations of X is denoted χ_X.

Given $X, Y \in \chi$ we can define $X \vee Y$ and $X \wedge Y$ in terms of the partial order $\leq$. More precisely, $X \vee Y$ is their *supremum*, i.e. the smallest measure of which both X and Y are subpopulations. In turn, $X \wedge Y$, the *infimum*, is the largest measure in $\chi_X \cap \chi_Y$. These operations can in turn be defined for any pair of subsets of subpopulations of Θ. The next lemma indicates that suprema and infima exists for all pairs of subsets of χ.

Lemma 1 *The partially ordered set $(\chi, \leq)$ is a complete lattice*

Proof. See [3]. ■

Let us denote the space of all finite and signed measures of bounded variation on $(\Theta, \sigma(\Theta))$ by $\mathcal{M}$. The norm on $\mathcal{M}$ is the total variation norm, that is, for $N \in \mathcal{M}$, $\|N\| = \sup \sum_{i=1}^{n} |N(S_i)|$, where the supremum is taken over all finite sequences (S_i) which are a partition of $\sigma(\Theta)$ ([5], Definition 4, pg. 97). Furthermore, this space is isometrically isomorphic to the topological dual of $\mathcal{C}(\Theta)$, the space of continuous functions over Θ ([5], Theorem 3, p. 265). So, we endow this space with the weak*-topology $\sigma(\mathcal{M}, \mathcal{C}(\Theta))$. Since χ is a weak*-closed subset of $\mathcal{M}$, we can endow it with the relative topology.

We now describe the preferences of both workers and firms. For the former case we consider the set of possible preferences denoted by $\mathcal{P}$. We assume that each worker has a strict preference over $\widetilde{F}$ represented by a bijection $P \in \mathcal{P}$, $P : \{1, 2, ...\} \mapsto \widetilde{F}$, yielding a linear ordering of $\widetilde{F}$. $P(j)$ will denote the identity of the worker´s j-th best alternative for $j = 1, 2, ...$

We shall denote the strict preference of f to f' according to $P \in \mathcal{P}$ by $f \succ_P f'$. For each $P \in \mathcal{P}$, we denote the set of all workers whose preferences are given by P as $\Theta_P \subset \Theta$. We shall assume that Θ_P belongs to $\sigma(\Theta)$ for every

[1] Our notation follows closely that in [3].

[2] i.e. $X(S) \leq G(S)$ for all $S \in \sigma(\Theta)$

$P \in \mathcal{P}$ and that the boundary of Θ_P has a null measure, that is, $G(\partial\Theta_P) = 0$.[3] Since all worker types have strict preferences, $\Theta_{P'} \cap \Theta_P = \emptyset$ for all $P, P' \in \mathcal{P}$ and $P \neq P'$. Thus $\Theta = \underset{P\in\mathcal{P}}{\cup} \Theta_P$

With respect to the preference of a given firm, we shall describe it indirectly by a correspondence $C_f : \chi \longrightarrow \chi$ such that $C_f(X) \subset \chi_X$ is the set of subpopulations of $X \in \chi$ that are the most preferred by f among all subpopulations of X. For completeness, we let $C_\emptyset(X) = \{X\}$ for all $X \in \chi$. This definition encompasses both the cases in which the firm is indifferent between alternatives and those in which choices are unique.

The resulting matching model is thus fully described by $(G, \widetilde{F}, (\Theta_P)_{P\in\mathcal{P}}, (C_f)_{f\in\widetilde{F}})$. We note that the product space $\chi^{|\widetilde{F}|}$ is a subset of the topological space $\mathcal{M}^{|\widetilde{F}|}$. We endow this space with the sup-norm inherited from $\mathcal{M}^{|\widetilde{F}|}$, i.e., for every $N = (N_f)_{f\in\widetilde{F}} \in \chi^{|\widetilde{F}|}$, $\|N\| = \sup_{f\in\widetilde{F}}\{\|N_f\| : N_f \in \chi\}$. Since $\mathcal{M} = \mathcal{C}(\Theta)^*$, we have that $\sigma(\mathcal{C}(\Theta), \mathcal{M})^{|\widetilde{F}|} = \sigma(\mathcal{C}(\Theta)^{|\widetilde{F}|}, \mathcal{M}^{|\widetilde{F}|})$ ([22], Theorem 4.3, 1. p. 137).

3 Stable matchings

Without loss of generality we conceive a matching as a collection of measures $M = (M_f)_{f\in\widetilde{F}}$ belonging to $\chi^{|\widetilde{F}|}$ such that $\sum_{f\in\widetilde{F}} M_f = G$.

Matchings can be ordered according to the criterion first introduced by Blair [2]. That is, given two matchings M and M' and $f \in \widetilde{F}$ we say that $M_f \succeq M'_f$, i.e., M_f is preferred to M'_f, if $M_f \in C_f\left(M_f \vee M'_f\right)$. In turn $M_f \succ M'_f$ (i.e., M_f is strictly preferred to M'_f) if $M_f \in C_f\left(M_f \vee M'_f\right)$ and $M'_f \notin C_f\left(M_f \vee M'_f\right)$. Let $F' \subset \widetilde{F}$, we shall say that $M' \succeq_{F'} M$ ($M' \succ_{F'} M$) if $M'_f \succeq M_f$ ($M'_f \succ M_f$) for all $f \in F'$. Let us introduce the following two measures.

$$D^{\succeq f}(M) := \sum_{P\in\mathcal{P}} \sum_{f'\in\widetilde{F}, f'\succeq_P f} M_{f'}(\Theta_P \cap \cdot)$$

and

$$D^{\preceq f}(M) := \sum_{P\in\mathcal{P}} \sum_{f'\in\widetilde{F}, f'\preceq_P f} M_{f'}(\Theta_P \cap \cdot)$$

The measure $M_{f'}(\Theta_P \cap \cdot)$ takes the value $M_{f'}(\Theta_P \cap S)$ for every $S \in \sigma(\Theta)$. We point out that these measures are well defined since every M_f is $\sigma-$finite and $\sum_{f\in\widetilde{F}} M_f$ is positive and upper bounded by G. $D^{\succeq f}(M)$ denotes the measure of workers that are assigned to firm f or better according to their preferences. On the other side, $D^{\preceq f}(M)$ weights the set of workers who are available for being rematched to firm f (or better) since it measures those assigned to firm

[3] As pointed out in [3] it is a technical assumption that makes simpler the analysis. Nevertheless it is satisfied when it is assumed that almost all agents have strict preferences which is a standard assumption in the literature or when the set Θ is discrete

f or worse according to their preferences. We shall say that $M \succeq_\Theta M'$ if $D^{\succeq f}(M) \geq D^{\succeq f}(M')$ for all $f \in \widetilde{F}$. We can thus say that the overall welfare of the workers is higher in M than in M' if for every firm f, the measure of workers assigned to f or better is larger in M than in M'

With these elements in hand we can define what a *stable* matching is:

Definition 2 *A matching $M = (M_f)_{f \in \widetilde{F}}$ is stable if*

1. *(a) For all $P \in \mathcal{P}$ and all $f \in \widetilde{F}$ if $\emptyset \succ_P f$ then $M_f(\Theta_P) = 0$.*

   (b) For every $f \in \widetilde{F}$, $M_f \in C_f(M_f)$.

2. *There is no $f \in F$ and $M'_f \in \chi$ such that $M'_f \leq D^{\preceq f}(M)$ and $M'_f \succ M_f$*

Here 1 is the so-called *individual rationality* condition while 2 is the *no blocking coalition* condition. They are known to be satisfied in models with one infinite side of the market ([3]). For our setting we add an additional definition:

Definition 3 *A matching M is* Pareto efficient *if there is no matching $M' \neq M$ and $F \subset \widetilde{F}$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$. It is* weakly Pareto efficient *if there is no matching $M' \neq M$ and $F \subset \widetilde{F}$ such that $M' \succ_F M$ and $M' \succ_\Theta M$*

Pareto efficiency means that there is no alternative matching in which the overall welfare of the workers is higher and the firms are at least so well as in M. Weak Pareto efficiency adds that there are no subset of firms and workers which are strictly better in M'.

4 Main results

Before stating the main theorem of this paper let us add two important assumptions on the preferences of the firms. Namely, for every $f \in \widetilde{F}$

1. C_f is a nonempty and a convex-valued correspondence and satisfies the revealed preference property: for any $X, X' \in \chi$ with $X' \leq X$, if $C_f(X) \cap \chi_{X'} \neq \emptyset$, then $C_f(X') = C_f(X) \cap \chi_{X'}$.
2. C_f is weak*-closed, that is, for any sequence (X^n, Y^n) converging to (X, Y) in the product of weak* topologies, such that $Y^n \in C_f(X^n)$ for every n, then $Y \in C_f(X)$.

Under these assumptions we have:

Theorem 4 *$(G, \widetilde{F}, (\Theta_P)_{P \in \mathcal{P}}, (C_f)_{f \in \widetilde{F}})$ has stable matchings.*

and

Proposition 5 *If a matching is stable then it is weakly Pareto efficient. If $|C_f(X)| = 1$ for each $f \in \widetilde{F}$, then it is Pareto efficient.*

Both results are proven in Appendix. We point out that the above assumptions are also used in [3]. The revealed preference property is well known in the matching literature. We refer to [13] among others. Regarding the continuity of the choice function it is a technical assumption which means that C_f has a closed graph. It is worth noticing that this is a very standard condition in many fields of economic theory as, for instance, general equilibrium theory with infinitely many commodities.

5 Discussion

In this paper we extended the many-to-one matching model to cover a market with both infinite workers and firms, although the cardinalities of the two sides are different (uncountable the former, countable the latter). We have shown that stable matchings exist and, furthermore, they are Pareto optimal. This is quite similar to the case of large economies in general equilibrium models.

Unsurprisingly, the methods applied to prove our main claims stem from the literature on infinite dimensional economies, in particular profiting from topological dualities. In our case, the duality between $\mathcal{M}$ and $\mathcal{C}(\Theta)$. Beyond this technical point, our assumptions over the preferences of the firms, namely that each C_f satisfies two conditions akin to continuity and the revealed preferences property lead quite naturally to the existence of stable matchings.

Further work involves extending the model to the case with uncountably infinite workers *and* firms in a non-seller/buyer settings. One possibility amounts to conceive the case presented here as involving not a countable number of firms but of *types* of firms of which there exists an uncountable infinite number.

A further, more complicated, extension is to the many-to-many case, which might require different topological conditions, leading to a more symmetrical treatment of the sets of workers and firms.

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6 Appendix

6.1 Proof of Theorem 4

We now turn to the proof of the theorem. The strategy is to construct a sequence of large economies with finitely many firms such that in every case there is a stable matching. Then, by a limit argument we obtain a stable matching for the original economy. In doing so, we structure the proof in three parts: First we define a sequence of truncated versions of the original economy. Second we show that each of them has a stable matching and third we study the limit point of the ensuing sequence of stable matchings and show it to be a stable matching of the original economy.

First part: Specification of a sequence of economies with finitely many firms.

For every natural number n, we define the set $F^n := \{f_1, f_2, ..., f_n\}$ and $\widetilde{F}^n := F^n \cup \{\emptyset\}$. It is clear that $\widetilde{F}^n \subset \widetilde{F}^{n+1} \subset \widetilde{F}^{n+2} \subset ...$, $\widetilde{F} = \bigcup_{n=1}^{\infty} \widetilde{F}^n$ and $|\widetilde{F}^n| = n+1$. The correspondence $C_f : \chi \longrightarrow \chi$ is that of Section 2 for $f \in \widetilde{F}^n$.

The preference of the workers are given by the restricted preference $P_{|\widetilde{F}^n} : \{1, ..., n+1\} \mapsto \widetilde{F}^n$. That is to say $P_{|\widetilde{F}^n}$ and P order the firms in $\widetilde{F}^n$ in the same way.

For every $f \in \widetilde{F}^n$ the measures $D^{n,\succeq f}(M)$ and $D^{n,\preceq f}(M)$ are defined as

$$D^{n,\succeq f}(M) := \sum_{P \in \mathcal{P}} \sum_{f' \in \widetilde{F}^n, f' \succeq_P f} M_{f'}(\Theta_P \cap \cdot)$$

and

$$D^{n,\preceq f}(M) := \sum_{P \in \mathcal{P}} \sum_{f' \in \widetilde{F}^n, f' \preceq_P f} M_{f'}(\Theta_P \cap \cdot)$$

.

Hence, the truncated large economy is specified by $(G, \widetilde{F}^n, (\Theta_P)_{P \in \mathcal{P}}, (C_f)_{f \in \widetilde{F}^n})$ and we obtain a sequence of large economies with finite sets of firms directed by inclusion $\{G, \widetilde{F}^n, (\Theta_P)_{P \in \mathcal{P}}, (C_f)_{f \in \widetilde{F}^n}\}_{n \geq 1}$.

Second part: Each of the terms of the sequence has a stable matching.

It is easy to check that Assumptions 1 and 2 hold in every economy $(G, \widetilde{F}_n, (\Theta_P)_{P\in\mathcal{P}}, (C_f)_{f\in\widetilde{F}^n})$. Consequently, by Theorem 2 of [3], there exists a stable matching $M^n = \left(M_f^n\right)_{f\in\widetilde{F}^n} \in \chi^{n+1}$ in every $(G, \widetilde{F}^n, (\Theta_P)_{P\in\mathcal{P}}, (C_f)_{f\in\widetilde{F}^n})$[4]. We expand it to the product space $\chi^{|\widetilde{F}|}$ while still writing it as $M^n = \left(M_f^n\right)_{f\in\widetilde{F}}$.[5] Let us recall that $\chi^{|\widetilde{F}|}$ is a subset of the topological dual of $\mathcal{C}(\Theta)^{|\widetilde{F}|}$ given by $\mathcal{M}^{|\widetilde{F}|}$.

Let us consider the set $\left\{(N_f)_{f\in\widetilde{F}} \in \mathcal{M}^{|\widetilde{F}|} : \sum_{f\in\widetilde{F}} T_f = G\right\}$. It can be easily checked that it is norm-bounded and by Alaoglu's Theorem ([21]) it is weak*-compact. Since M^n belongs to BM for every n, there exists a subsequence $(M^{n_k})_{k\in K}$, where K is a subset of the natural numbers, converging to M in the weak*-topology. Since a sequence in a product space converges if and only if the projection of each component converges ([16], p. 91), we have a limit $M = (M_f)_{f\in\widetilde{F}}$.

Third part: M is a stable matching.

Since for every $k \in K$, $\sum_{f\in\widetilde{F}} M_f^{n_k} = G$ and χ is weak*-closed, we have that $\sum_{f\in\widetilde{F}} M_f = G$, hence M is a matching. We now show that it is stable:

(i) Let $f \in \widetilde{F}$ and $P \in \mathcal{P}$ such that $\emptyset \succ_P f$. There exists n_0 such that $f \in \widetilde{F}^{n_0}$. There exists $k_0 \in K$ such that for all $k \geq k_0$, $\widetilde{F}^{n_0} \subset \widetilde{F}^{n_k}$. Since $M^{n_k} = \left(M_f^{n_k}\right)_{f\in\widetilde{F}}$ is a stable matching we have that $M_f^{n_k}(\Theta_P) = 0$ for all $k \geq k_0$. Hence, $M_f(\Theta_P) = 0$.

(ii) Let $f \in \widetilde{F}$. There exists n_0 such that $f \in \widetilde{F}^{n_0}$. There exists $k_0 \in K$ such that for all $k \geq k_0$, $\widetilde{F}^{n_0} \subset \widetilde{F}^{n_k}$. Since $M^{n_k} = \left(M_f^{n_k}\right)_{f\in\widetilde{F}}$ is a stable matching we have that $M_f^{n_k} \in C_f\left(M_f^{n_k}\right)$ for all $k \in K$. By Assumption 2., $M_f \in C_f(M_f)$.

We have proven that the matching M satisfies individual rationality. It only remains to show that there is no blocking coalition. We proceed by contradiction. Suppose that there exists $f \in F$ and $M'_f \in \chi$ such that $M'_f \leq D^{\preceq f}(M)$, $M'_f \in C_f\left(M_f \vee M'_f\right)$ and $M_f \notin C_f\left(M_f \vee M'_f\right)$. It is clear that $M_f \leq M_f \vee M'_f$, whence $M_f \in \chi_{M_f \vee M'_f}$. By (ii) above, $M_f \in C_f(M_f)$ and hence, $M_f \in C_f(M_f) \cap \chi_{M_f\vee M'_f}$. Consequently, by the revealed preference condition, $M_f \in C_f\left(M_f \vee M'_f\right)$, a contradiction.

[4]We remark that Theorem 2 in [3] is proved under the assumption that $\mathcal{P}$ is finite. Nevertheless, the proof they provide also works for the infinite case.

[5]i.e. we add the null measure O_f^n to the product spaces in the following way: $M^n = \left(M_f^n\right)_{f\in\widetilde{F}^n} \times \left(O_f^n\right)_{f\in\widetilde{F}\setminus\widetilde{F}^n}$

6.2 Proof of Proposition 5

First of all, we claim that[6]

$$\sum_{P\in\mathcal{P}} \sum_{f'\in\widetilde{F}, f'\succeq_P \bar{f}} M'_{f'}(\Theta_P\cap S) \geq \sum_{P\in\mathcal{P}} \sum_{f'\in\widetilde{F}, f'\succeq_P \bar{f}} M_{f'}(\Theta_P\cap S) \tag{1}$$

for all $S\in\sigma(\Theta)$ and all $\bar{f}\in\widetilde{F}$ is equivalent to saying that

$$\sum_{f'\in\widetilde{F}, f'\succeq_P \bar{f}} M'_{f'}(\Theta_P\cap S) \geq \sum_{f'\in\widetilde{F}, f'\succeq_P \bar{f}} M_{f'}(\Theta_P\cap S) \tag{2}$$

for all $S\in\sigma(\Theta)$, for all $\bar{f}\in\widetilde{F}$ and all $P\in\mathcal{P}$. Indeed, for $S\in\sigma(\Theta)$, one has that $S\cap\Theta_P$ belongs to $\sigma(\Theta)$ since Θ_P is assumed measurable. Then, since $S\cap\Theta_P$ does not meet $\Theta_{P'}$ for $P'\neq P$ it follows that (1) implies (2). The converse is immediate.

We proceed by contradiction. Let us suppose that there exists M' such that $M'\succ_F M$ and $M'\succ_\Theta M$. Consequently there exists $f\in F$ such that $M'_f\succ M_f$. For a given preference $P\in\mathcal{P}$ we denote the immediate predecessor of f as f_-^P.[7]. Since $M'\succ_\Theta M$ then, for each $\bar{f}\in\widetilde{F}$, one has

$$\sum_{f'\in\widetilde{F}, f'\succeq_P \bar{f}} M'_{f'}(\Theta_P\cap S) \geq \sum_{f'\in\widetilde{F}, f'\succeq_P \bar{f}} M_{f'}(\Theta_P\cap S)$$

for all $S\in\sigma(\Theta)$ and $P\in\mathcal{P}$. In particular for f_-^P

$$\sum_{f'\in\widetilde{F}, f'\succeq_P f_-^P} M'_{f'}(\Theta_P\cap S) \geq \sum_{f'\in\widetilde{F}, f'\succeq_P f_-^P} M_{f'}(\Theta_P\cap S)$$

for all $S\in\sigma(\Theta)$

Which is equivalent to

$$\sum_{f'\in\widetilde{F}, f'\succ_P f} M'_{f'}(\Theta_P\cap S) \geq \sum_{f'\in\widetilde{F}, f'\succ_P f} M_{f'}(\Theta_P\cap S) \tag{3}$$

for all $S\in\sigma(\Theta)$

Given $P\in\mathcal{P}$ we have

$$\sum_{f\in\widetilde{F}} M'_f(\Theta_P\cap S) = G(\Theta_P\cap S) = \sum_{f\in\widetilde{F}} M_f(\Theta_P\cap S) \text{ for all } S\in\sigma(\Theta)$$

which can be rewritten as

$$\sum_{f'\in\widetilde{F}, f'\succ f} M'_{f'}(\Theta_P\cap S) + \sum_{f\in\widetilde{F}, f'\preceq f} M'_{f'}(\Theta_P\cap S) = \sum_{f'\in\widetilde{F}, f'\succ f} M_{f'}(\Theta_P\cap S) + \sum_{f'\in\widetilde{F}, f'\preceq f} M_{f'}(\Theta_P\cap S)$$

[6]We thank a clarification of Fuhito Kojima which simplified our proof

[7]It means that $f_-^P\succ f$ and if $f'\succ f$ then $f'\succeq f_-^P$

for all $S \in \sigma(\Theta)$.

We claim that $\sum\limits_{f \in \widetilde{F}, f' \preceq f} M'_{f'}(\Theta_P \cap S) \leq \sum\limits_{f' \in \widetilde{F}, f' \preceq f} M_{f'}(\Theta_P \cap S)$ for all $S \in \sigma(\Theta)$. Otherwise, from the previous equality one deduces that $\sum\limits_{f' \in \widetilde{F}, f' \succ f} M'_{f'}(\Theta_P \cap S) < \sum\limits_{f' \in \widetilde{F}, f' \succ f} M_{f'}(\Theta_P \cap S)$ for some $S \in \sigma(\Theta)$ which contradicts (3). Summing over $\mathcal{P}$,

$$\sum_{P \in \mathcal{P}} \sum_{f \in \widetilde{F}, f' \preceq f} M'_{f'}(\Theta_P \cap S) \leq \sum_{P \in \mathcal{P}} \sum_{f' \in \widetilde{F}, f' \preceq f} M_{f'}(\Theta_P \cap S)$$

for all $S \in \sigma(\Theta)$

Since $M'_f \succ M_f$ and $M'_f \leq D^{\preceq f}(M')$ we have a contradiction with the fact that M is a stable matching. Consequently, M is weakly Pareto efficient.

Now, if $|C_f(X)| = 1$ for all $X \in \chi$ and we assume that M is not Pareto efficient, then there is another matching M' and $F \subset \widetilde{F}$ such that $M' \neq M$, $M' \succeq_F M$ and $M' \succeq_\Theta M$. Consequently, there exists $f \in F$ such that $M'_f \succeq M_f$ which means that $M'_f = C_f(M'_f \vee M_f)$. Then $M_f \neq M'_f$, whence $M'_f \succ M_f$ which contradicts the fact that M is a stable matching.